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Thermal instabilities in rapidly rotating systems

By F. H. BUSSE
Max-Planck-Institut für Physik und Astrophysik, Munich

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Thermal instabilities of a contained fluid are investigated for a fairly general class of problems in which the dynamics are dominated by the effects of rotation. In systems of constant depth in the direction of the axis of rotation the instability develops when the buoyancy forces suffice to overcome the stabilizing effects of thermal conduction and of viscous dissipation in the Ekman boundary layers. Owing to the Taylor–Proudman theorem, a slight gradient in depth exerts a strongly stabilizing influence. The theory is applied to describe the instability of the 'lower symmetric régime' in the rotating annulus experiments at high rotation rates. An example of geophysical relevance is the instability of a self-gravitating, internally heated, rotating fluid sphere. The results of the perturbation theory for this problem agree reasonably well with the results of an extension of the analysis by Roberts (1968).

1. Introduction

Buoyancy-driven convective motions caused by an inhomogeneous distribution of heat sources and sinks are a common phenomenon in the fields of geophysics and astrophysics. Their dynamics are often strongly dependent on the fact that they occur in rotating systems. In order to simulate the convective phenomena on a laboratory scale, a number of experiments have been undertaken in recent years. An example is the rotating annulus experiment, which was investigated by Hide (1958) and others. A review of the work on this and related topics has been given by Fultz (1961).

The goal of the experimental studies motivated primarily by meteorological problems has been the investigation of baroclinic instabilities. These instabilities are driven by the release of gravitational energy, which is available because the gravity vector and the temperature gradient of the basic state do not coincide. The scalar product between the two vectors, however, has always a negative sign, and the stratification is an important ingredient of the dynamics of the baroclinic waves. In contrast, the convective motions considered here occur as instabilities of basic states for which the scalar product between temperature gradient and gravity vector (including the centrifugal force) is positive. A more direct release of gravitational energy is possible. Therefore, the instability occurs at a lower value of the destabilizing temperature gradient than is the case of baroclinic waves.

A second distinctive feature of the problems to be discussed here is the property that the axis of rotation and the direction of the buoyancy force do not coincide.

This property is characteristic for problems of convection in the liquid core of the earth and in stars, which have motivated the present study. In the above-mentioned experiments, this situation is usually not realized, except when the centrifugal force becomes comparable to gravity. For this reason, the analysis is applicable to the annulus experiment only in the case of high rotation rates when the potential surfaces are no longer perpendicular to the rotation axis.

The theory of the convective motions starts from the fact that they can be characterized as instabilities of a 'symmetric' state. The symmetric state is described by the solution of the basic equations that reflects all symmetries of the physical conditions defining the problem. This is the unique solution for sufficiently low values of a relevant parameter, such as, for example, the Rayleigh parameter, which is proportional to a characteristic temperature difference and inversely proportional to the diffusivities of heat and momentum. In special cases the symmetric solution corresponds to a vanishing velocity field.

When the Rayleigh parameter exceeds a finite critical value, the symmetric state becomes unstable. Convective motions appear which are characterized by a wavelength in the dimension with respect to which the symmetric state is invariant. Instabilities of this kind will be analyzed here on the assumption that they can be regarded as disturbances of infinitesimal amplitude. We shall base the discussion on the Boussinesq approximation of the basic equations, and restrict it to the case of a rapidly rotating system, in which the Coriolis force by far exceeds the destabilizing component of the buoyancy force.

The latter restriction allows us to take advantage of two properties of the system. First, the Taylor–Proudman theorem applies, since, in nearly stationary flow, the Coriolis force can be balanced only by the pressure gradient. The geostrophic balance provides a strong restriction, yet does not determine the disturbances. For this purpose, buoyancy and viscous forces have to be introduced as perturbations. The solvability condition for the linear inhomogeneous perturbation equation then supplies the necessary information for the complete determination of the disturbances and of the critical parameter value at which they first become possible. The second property is the fact that in many cases the viscous dissipation is localized in the Ekman boundary layers close to the rigid walls. The thickness of the Ekman layers is of the order $E^{\frac{1}{2}}L$, where L represents the depth of the system in the direction of the axis of rotation, and the Ekman number E is defined by

$$E = \nu/\Omega L^2.$$

Ω is the constant rotation rate of the system and ν denotes the kinematic viscosity. Since $E^{\frac{1}{2}}$ is small compared with unity in most cases, solutions can be obtained in terms of a boundary-layer approximation.

The theoretical analysis in §2–§5 is intended to exhibit characteristic features of the thermal instability, rather than provide a quantitative comparison with particular experiments. For this reason the discussion proceeds by avoiding lengthy calculations, and by considering the simplest geometry for a particular problem. The notation is chosen in such a way that the treatment can be generalized easily to fit more complex problems.

In §2 the thermal instability in an annulus rotating about its vertical axis is considered in the case when the outer side wall is maintained at a higher temperature than the inner side wall. We first assume that the fluid has a uniform depth. The mathematical analysis is similar to the treatment of shear flow instabilities in rotating systems (Busse 1968): it was indicated there that a slightly changing depth of the fluid has a significant influence on the instability. As an example of this effect in the case of thermal instabilities, the annulus with a free surface will be considered in §3. Since the Taylor–Proudman theorem allows only deviations of higher order from the two-dimensional form of the disturbance, the horizontal scale of the critical disturbance has to decrease in order to minimize the constraint of the changing depth. This tendency is balanced by the increasing importance of the viscous forces in the interior of the fluid. The transition of the 'lower symmetric' régime at large values of Ω in the annulus experiment of Fowlis & Hide (1965) corresponds to this case.

Rapidly rotating stars and the core of the earth are examples of fluid systems in which convective motions subjected to the constraint of a changing depth may occur. The thermal instability of a uniformly heated self-gravitating fluid sphere has attracted particular attention as a model for convection in the earth core. The most detailed analysis of this problem has been given by Roberts (1968). A comparison of Robert's numerical results with the conclusions of the perturbation theory in §4 suggests that modes of different symmetry than those considered by Roberts should lead to a lower critical Rayleigh number. For this reason the problem is re-analyzed in §5, with the effect that the results of the exact theory and the perturbation theory are found to be in reasonable agreement.

2. Instabilities in systems of uniform depth

In §2 a fluid filled annulus which rotates about its axis of symmetry with the constant angular velocity Ω is considered. The direction of the rotation vector is opposite to that of the force of gravity, and is denoted by the unit vector $\mathbf{k}$. We shall use the small-gap approximation, i.e. we asume that the distance D between the cylindrical side walls is small compared to the mean radius r_0 of the annulus. This permits us to neglect the effects of curvature, and to use a Cartesian system of co-ordinates. The fluid in the annular channel is bounded at top and bottom by horizontal rigid boundaries. Accordingly, the depth of the fluid L is uniform. We introduce a dimensionless description by using L, Ω^{-1}, and $(T_2-T_1)\,\Omega L^3/D\kappa$ as scales for length, time, and temperature, respectively. T_1 and T_2 are the given fixed values of the temperature on the inner and the outer wall of the annular channel. κ is the thermometric conductivity. The Boussinesq equations of motion for the velocity vector $\hat{\mathbf{v}}$ and the heat equation for the temperature $\hat{\vartheta}$ are

$$E\nabla^2\hat{\mathbf{v}} - \nabla\hat{p} - RE^{\frac{1}{2}}\left(\mathbf{i} - \mathbf{k}\frac{g}{\Omega^2 r_0}\right)\hat{\vartheta} = 2\mathbf{k}\times\hat{\mathbf{v}} + \hat{\mathbf{v}}\,.\,\nabla\hat{\mathbf{v}} + \frac{\partial}{\partial t}\hat{\mathbf{v}}, \tag{2.1}$$

$$\nabla\,.\,\hat{\mathbf{v}} = 0, \tag{2.2}$$

$$\frac{\kappa}{\Omega L^2}\nabla^2\hat{\vartheta} = \hat{\mathbf{v}}\,.\,\nabla\hat{\vartheta} + \frac{\partial}{\partial t}\hat{\vartheta}. \tag{2.3}$$

The origin of the Cartesian system of co-ordinates is located in the centre of the channel, with the x, y, z co-ordinates in the directions of the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, respectively, where $\mathbf{i}$ represents the direction of the centrifugal force.

Since we expect a balance between the buoyancy force and the viscous stresses in the Ekman layer, we have introduced the parameter

$$R = \frac{\alpha(T_2 - T_1)\,\Omega r_0 L^2}{\kappa E^{\frac{1}{2}} D}, \tag{2.4}$$

which we call the Rayleigh parameter. The viscosity enters this expression in the form of the Ekman number $E = \nu/\Omega L^2$. α is the expansion coefficient. The definition (2.4) has been motivated by the analogy with the case of convection in a layer heated from below. Gravity and kinematic viscosity in the usual definition of the Rayleigh number are replaced in expression (2.4) by $\Omega^2 r_0$ and $E^{\frac{1}{2}}\Omega L^{-2}$, respectively. In adition to R and E, the parameters $g/\Omega^2 r_0$ and $\kappa/\Omega L^2$ appear in the equations. No particular symbols have been introduced for them, since they will drop out of the final result of § 2. We require only that they satisfy the not very restrictive conditions

$$g/\Omega^2 r_0 \ll E^{-\frac{1}{2}}, \quad \kappa/\Omega L^2 \ll E^{-\frac{1}{2}}.$$

The parameter $\kappa/\Omega L^2$ will, in fact, be of order E, unless a fluid with a very small Prandtl number

$$P \equiv \nu/\kappa$$

is considered. By leaving the order of $\kappa/\Omega L^2$ unspecified we include the latter possibility.

Equations (1.1)–(1.3) are satisfied exactly by the following solution (which is axisymmetric, i.e. the y-dependence vanishes):

$$\hat{\mathbf{v}} = \mathbf{U} \equiv \mathbf{j}(z+c)\,\tfrac{1}{2}RE^{\frac{1}{2}}\frac{g}{\Omega^2 r_0}\frac{\kappa}{\Omega L^2}, \tag{2.5a}$$

$$\hat{\vartheta} = \Theta \equiv x\frac{\kappa}{\Omega L^2} + \frac{T_2 - T_1}{2(T_2 - T_1)}. \tag{2.5b}$$

This solution shows that a purely conductive solution is possible, even though gravity and temperature gradient to not coincide in contrast to a non-rotating system. In a meteorological context, the velocity field (2.5a) is called the thermal wind. The solution (2.5) satisfies the boundary condition for the temperature,

$$\left.\begin{aligned} \hat{\vartheta} - \frac{T_2 + T_1}{2(T_2 - T_1)} = \pm\frac{1}{2}\frac{D\kappa}{\Omega L^3} \quad \text{at} \quad x = \pm\frac{1}{2}\frac{D}{L}, \\ \partial_z\hat{\vartheta} = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}, \end{aligned}\right\} \tag{2.6}$$

where top and bottom have been assumed as thermal insulators. The solution (2.5) however, does not satisfy the boundary condition for the velocity field,

$$\left.\begin{aligned} \hat{\mathbf{v}} = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}(D/L), \\ \hat{\mathbf{v}} = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \end{aligned}\right\} \tag{2.7}$$

It can be assumed, nevertheless, that (2.5) describes the symmetric state correctly in the interior of the channel, and that the boundary-layer regions in which the solution has to be modified become vanishingly small in the limit, when E tends to zero. Hunter (1967), extending the earlier work of Robinson (1959), has given a detailed analysis to prove this point. He shows that, in the limit

$$RE\,\frac{Dg}{L\Omega^2 r_0} \ll 1, \tag{2.8}$$

the solution (2.5) is established throughout the interior. In the case of a rigid top surface as assumed in (2.7), the constant c of integration in (2.5a) vanishes; it is equal to $\frac{1}{2}$ in the case of a free surface.

In order to analyze the stability of the symmetric state, we superimpose disturbances of infinitesimal amplitude on the solution (2.5). By setting

$$\hat{\mathbf{v}} = \mathbf{U}+\mathbf{v}, \quad \hat{\vartheta} = \Theta+\vartheta,$$

we obtain the following equations for $\mathbf{v}$ and ϑ from (2.1) and (2.3):

$$E\nabla^2\mathbf{v}-\nabla p - RE^{\frac{1}{2}}\left(\mathbf{i}-\mathbf{k}\,\frac{g}{\Omega^2 r_0}\right)\vartheta = 2\mathbf{k}\times\mathbf{v}+\mathbf{U}\,.\,\nabla\mathbf{v}+\mathbf{v}\,.\,\nabla\mathbf{U}+\frac{\partial}{\partial t}\mathbf{v}, \tag{2.9}$$

$$\nabla^2\vartheta = \mathbf{v}\,.\,\mathbf{i}+\frac{L^2\Omega}{\kappa}\left(\mathbf{U}\,.\,\nabla\vartheta+\frac{\partial}{\partial t}\vartheta\right). \tag{2.10}$$

Since we are interested in the solution of the problem for small values of $E^{\frac{1}{2}}$, we introduce an expansion in powers of $E^{\frac{1}{2}}$ for $\mathbf{v}$,

$$\mathbf{v} = \mathbf{v}_0+E^{\frac{1}{2}}\mathbf{v}_1+\ldots+\text{b.l.m.}, \tag{2.11}$$

and an analogous expansion for ϑ. We assume that the functions $\mathbf{v}_n$ describe the velocity field in the interior. Close to the boundaries, a boundary-layer modification (b.l.m.) has to be added to the interior velocity field. We do not have to consider this modification explicitly. It can be shown (see Greenspan 1968) that the total velocity field satisfies the boundary condition $\mathbf{v} = 0$ (at least to order $E^{\frac{1}{2}}$), when the interior velocity field satisfies the boundary condition,

$$\mathbf{n}\,.\,\mathbf{v} = -\tfrac{1}{2}E^{\frac{1}{2}}\mathbf{n}\,.\,\nabla\times\left\{\left(\mathbf{n}\times\mathbf{v}+\frac{\mathbf{n}\,.\,\mathbf{k}}{|\mathbf{n}\,.\,\mathbf{k}|}\,\mathbf{v}\right)|\mathbf{n}\,.\,\mathbf{k}|^{-\frac{1}{2}}\right\}. \tag{2.12}$$

$\mathbf{n}$ denotes the normal unit vector of the bounding surface. The expression (2.12) diverges when the boundary is parallel to the axis of rotation. The thickness of the viscous boundary layer formed in this case is of order $E^{\frac{1}{3}}$. Hence, its contribution to the dissipation of the system is negligible compared to the dissipation in the Ekman layers. This justifies the use of the boundary condition,

$$\mathbf{n}\,.\,\mathbf{v} = 0, \tag{2.13}$$

at the side walls.

The derivation of the boundary condition (2.12) for the velocity field depends on the property that the time-dependence of the velocity field is small compared with 1, if not vanishing. The following analysis will be restricted to disturbances satisfying this property. Accordingly, we assume a time-dependence of the form,

$$\exp\{E^{\frac{1}{2}}\sigma t\}. \tag{2.14}$$

The discussion of disturbances with a time-dependence of order one is given in the appendix. It is shown there that the corresponding instabilities, which in lowest order can be described as inertial oscillations, lead in general to higher critical values of the Rayleigh parameter.

By introducing the expansion (2.11) in (2.9) we obtain as basic balance

$$\left.\begin{aligned} 2\mathbf{k}\times\mathbf{v}_0+\nabla p_0 &= 0,\\ \nabla.\mathbf{v}_0 &= 0.\end{aligned}\right\} \quad (2.15)$$

The general solution of these equations satisfying the condition $\mathbf{k}.\mathbf{v}_0 = 0$ at $z = \pm\frac{1}{2}$ is

$$\mathbf{v}_0 = \mathbf{k}\times\nabla\tfrac{1}{2}p_0, \quad (2.16)$$

where p_0 is an arbitrary function of x and y and independent of z. The boundary condition (2.13) at the side walls requires that p_0 vanishes at $x = \pm\frac{1}{2}D/L$.

In order to obtain additional information about p_0, we have to consider the equations for $\mathbf{v}_1$,

$$2\mathbf{k}\times\mathbf{v}_1+\nabla p_1 = -R\{\mathbf{i}-\mathbf{k}(g/\Omega^2 r_0)\}\vartheta_0-E^{-\frac{1}{2}}\mathbf{U}.\nabla\mathbf{v}_0-\sigma\mathbf{v}_0, \quad (2.17)$$

$$\nabla.\mathbf{v}_1 = 0. \quad (2.18)$$

Equations (2.17), (2.18), together with the boundary condition following from (2.12), represent a linear inhomogeneous boundary-value problem. The necessary and sufficient condition for the existence of a solution of the problem is that the inhomogeneity is orthogonal to each solution of the adjoint homogeneous problem. The homogeneous part of (2.17) represents an antisymmetric operator. Hence, the manifold of solutions of the adjoint homogeneous problem is identical with the manifold of solutions given by

$$\mathbf{v}^* = \mathbf{k}\times\nabla\tfrac{1}{2}p^*, \quad (2.19)$$

where p^* satisfies the same constraints as p_0. Multiplication of (2.17) by (2.19), and integration over the contained fluid, yields

$$\int \mathbf{v}_1\nabla p^* dV = \int \mathbf{k}\times\nabla\tfrac{1}{2}p^*.\mathbf{J}(\mathbf{v}_0)dV. \quad (2.20)$$

$\mathbf{J}(\mathbf{v}_0)$ stands for the right-hand side of (2.17). By transforming the left-hand side of (2.20) into a surface integral, and by partial integration of the right-hand side, we obtain

$$-\iint p^*\mathbf{k}.\nabla\times\mathbf{v}_0 dx dy = \iint p^*\left\{\tfrac{1}{2}\nabla.\mathbf{k}\times\int\mathbf{J}(\mathbf{v}_0)dz\right\}dx dy, \quad (2.21)$$

where $\mathbf{k}.\mathbf{v}_1$ has been eliminated according to the boundary condition (2.12). Since this solvability condition has to be satisfied for arbitrary functions p^* of x and y vanishing at $x = \pm\frac{1}{2}D/L$, the integrands on both sides of (2.21) must be equal,

$$\begin{aligned}\mathbf{k}.\nabla\times(\nabla\times\mathbf{k}\tfrac{1}{2}p_0) &= \tfrac{1}{2}\nabla.\mathbf{k}\times\textstyle\int\mathbf{J}(\mathbf{v}_0)dz,\\ &= -\tfrac{1}{2}R\mathbf{k}\times\mathbf{i}.\nabla\textstyle\int\vartheta_0 dz-\tfrac{1}{2}\sigma\nabla.\mathbf{k}\times\mathbf{v}_0.\end{aligned} \quad (2.22)$$

In the second line of this relation, some terms of $\mathbf{J}(\mathbf{v}_0)$ have disappeared, because they are parallel to $\mathbf{k}$ or their z-average vanishes. In order to determine the critical value of the Rayleigh parameter, we have to solve (2.22) in the case when the real part of σ vanishes. Since there is no term which could lead to an imaginary part, σ can be replaced by zero. To eliminate ϑ_0 from (2.22), we have to return to the heat equation (2.10), which in lowest order reduces to

$$\nabla^2 \vartheta_0 = \mathbf{i}.\mathbf{v}_0. \tag{2.23}$$

The boundary condition (2.6), and the fact that ϑ_0 does not depend on z according to (2.23), allow us to drop the integration symbol in (2.22) and to obtain as equation for p_0,

$$\Delta_2^2 p_0 = -\tfrac{1}{2} R \frac{\partial^2}{\partial y^2} p_0. \tag{2.24}$$

The operator Δ_2 is defined by

$$\Delta_2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The boundary conditions,

$$p_0 = \frac{\partial^2}{\partial x^2} p_0 = 0 \quad \text{at} \quad x = \pm \frac{1}{2}\frac{D}{L},$$

are satisfied by the following solution of (2.24):

$$p_0 = \exp(iay) \cos \pi (L/D) x, \tag{2.25}$$

with the corresponding eigenvalue,

$$R = 2[\{(L/D)\pi\}^2 + a^2]^2/a^2. \tag{2.26}$$

The minimum R_c of R as a function of a,

$$R_c = 8\left(\frac{L}{D}\pi\right)^2, \tag{2.27}$$

represents the critical value at which the symmetric state (2.5) becomes unstable.

We do not know of experimental observations which correspond to the theoretical problem discussed in §2. Data on the transition from the symmetric régime in an annulus are usually obtained in the case of a free fluid surface. The fact that this surface is inclined according to the relative magnitude of centrifugal force and gravity changes the problem dramatically. We shall return to this problem at the end of §3.

For completeness, we mention that we have assumed throughout §2 that D/L is of order one, i.e. that the inequalities,

$$E^{\frac{1}{2}} \ll L/D \ll E^{-\frac{1}{2}}, \tag{2.28}$$

are satisfied. The boundary-layer method breaks down if the first of these relations is violated. In the case of a very tall annulus, for which the second relation of (2.28) no longer holds, the problem becomes mathematically identical with the problem of Bénard convection in a layer heated from below. The centrifugal force replaces gravity in this case, and the effect of the Coriolis force on the critical value of the Rayleigh number vanishes.

Although the analysis of §2 has been restricted to a system in which top and bottom surface are perpendicular to the axis of rotation, it can be regarded as representative of general symmetric systems of constant depth. We refer to the analogous case of shear flow instability (Busse 1968), where it has been shown that the results are essentially identical when the distance between the two parallel surfaces is used in place of L.

3. Instabilities in systems of slightly changing depth

(i) It is known from other theories of rotating systems (see Greenspan 1968) that the geostrophic modes, which are stationary in a system of constant depth, correspond to the slowly drifting Rossby waves in a system with slightly changing depth. We expect accordingly that, in the latter kind of system, the imaginary part of σ does not vanish for the critical disturbance. For simplicity we consider the same geometry as in §2, with the difference that the normal vectors $\mathbf{n}^T$ and $\mathbf{n}^B$ of the top and bottom surface are slightly inclined with respect to the axis of rotation. The modification due to this effect is small. The lowest order of the solution is still governed by (2.15). Since the solution (2.16) does not completely satisfy the boundary condition $\mathbf{n}.\mathbf{v}_0$ at the top and bottom, the term

$$-\mathbf{n}\times\mathbf{k}.\nabla p_0/2$$

has to be added to the inhomogeneous part of the boundary condition for $\mathbf{n}.\mathbf{v}_1$. In place of (2.24) we obtain

$$(\Delta_2 - i\omega PE^{-\frac{1}{2}})\left\{\Delta_2(2+i\omega) - 4\eta E^{-\frac{1}{2}}\frac{\partial}{\partial y}\right\}p_0 = -R\frac{\partial^2}{\partial y^2}p_0, \tag{3.1}$$

where η is proportional to the change in depth,

$$\eta \equiv \tfrac{1}{2}\mathbf{i}.(\mathbf{n}^T + \mathbf{n}^B), \tag{3.2}$$

and σ has been set equal to $i\omega$.

Equation (3.1) and the corresponding boundary conditions are satisfied by solutions of the form (2.25), for which we obtain the following dispersion relation for ω:

$$\omega\{2PE^{-\frac{1}{2}} + (L\pi/D)^2 + a^2\} = -4\eta E^{-\frac{1}{2}}a. \tag{3.3}$$

With this relation inserted, the real part of (3.1) yields

$$\tfrac{1}{2}R = \left\{\left(\frac{\pi L}{D}\right)^2 + a^2\right\}^2 a^{-2} + \frac{(4\eta PE^{-1})^2}{\{2PE^{-\frac{1}{2}} + (L\pi/D)^2 + a^2\}^2} \tag{3.4}$$

as expression for the eigenvalue R.

The critical Rayleigh parameter R_c is determined by the minimum of (3.4) as a function of a. We note, without going into a detailed evaluation of (3.4), that a_c will be of order one as long as the relation

$$\eta \ll E^{\frac{1}{4}}P^{\frac{1}{2}} \tag{3.5}$$

holds. The corresponding value R_c will be of the same order of magnitude, as in the case of constant depth. The situation becomes more interesting when the relation (3.5) is not satisfied. In this case, the critical wave-number increases, and the viscous dissipation in the interior becomes important. The following second part of §3 discusses this problem.

(ii) We consider the case when η is large compared with $\sqrt{E}$ yet still small compared with 1. It is appropriate in this case to use η, in place of $\sqrt{E}$, as the expansion parameter in (2.9), and to introduce

$$R^* \equiv RE^{\frac{1}{2}}/\eta \tag{3.6}$$

as relevant Rayleigh parameter. The basic symmetric state will not be influenced very much by the changing depth. Thus, the analysis of the problem follows the steps described in § 2. Since we are interested in the case when the viscous forces in the interior become important, we neglect the dissipation in the Ekman layer for simplicity. The case when both dissipative effects are comparable can be treated in an analogous way. Accordingly, the solvability condition yields the following eigenvalue problem in place of (2.24):

$$(\Delta_2 - i\omega^*\eta E^{-1}P)\left\{(E\eta^{-1}\Delta_2 - i\omega^*)\Delta_2 + 4\frac{\partial}{\partial y}\right\}p_0 = R^*\frac{\partial^2}{\partial y^2}p_0. \tag{3.7}$$

The time-dependence of the marginal disturbance has been assumed in the form $\exp\{i\eta\omega^* t\}$. The y-dependence will be assumed again in the form $\exp\{iay\}$. Because no assumptions about the horizontal scale have been made *a priori*, all terms which may possibly become relevant in (3.7) have to be included. The consistency of the expansion has to be checked after the horizontal scale has been determined.

Since the viscous stresses in the interior have been taken into account the vanishing of the parallel component of the velocity at the side walls has to be required. For this reason, the x-dependence of the pressure p will differ from that assumed in (2.25). We expect, however, that the x-dependence will be negligible in comparison with the y-dependence in the case of sufficiently large values η in which we are interested. This expectation suggests that we should consider (3.7) as an algebraic equation with $-a^2$ in place of the operator Δ_2.

The imaginary part yields the dispersion relation,

$$(1+P)a^4\omega^* + 4a^3 = 0; \tag{3.8}$$

and the real part yields the following expression for R^*:

$$R^* = a^4\frac{E}{\eta} + \left(\frac{4P}{1+P}\right)^2\frac{\eta}{Ea^2}. \tag{3.9}$$

The critical values a_c and R_c^* are obtained by minimizing the right-hand side in this relation:

$$a_c = \left(\frac{4\eta P}{E(1+P)\sqrt{2}}\right)^{\frac{1}{3}}, \tag{3.10}$$

$$R_c^* = 3\left(\frac{\eta}{4E}\right)^{\frac{1}{3}}\left(\frac{4P}{1+P}\right)^{\frac{4}{3}}. \tag{3.11}$$

The result is consistent with the fact that it was derived by a perturbation analysis, starting with (2.15) as basic balance as long as $\sqrt{E}$ and η are small compared with 1. The dissipation in the Ekman layers is negligible if a^2 is large compared to $E^{-\frac{1}{2}}$, or if

$$\eta P \gg E^{\frac{1}{4}} \tag{3.12}$$

holds. That the x-dependence is in fact small compared with the y-dependence can be tested by assuming a dependence of the form $\cos a\gamma x$. It turns out that the minimum of R^* is reached for $\gamma \to 0$, which verifies our expectation.

The results of the two extreme cases treated in § 3, in which the viscous dissipation was located either in the Ekman layer or in the interior, are similar. The general case, in which both effects are present, will be more complex merely in the computation of the critical Rayleigh number. Owing to the change in depth, the instabilities always have the form of Rossby waves with the characteristic property that they propagate in the direction of either the positive or the negative y co-ordinate depending on the sign of η.

One of the motivations for the analysis in § 3 was to give a description of the instability in the case when the fluid in the annulus has a free surface. That the centrifugal force is responsible for the transition from the lower symmetric régime at large rotation rates is best seen in the paper by Fultz (1961), who compares the case of an annulus cooled from the inside and heated from the outside with the reverse case. The change in depth in this problem is

$$\eta = \frac{\Omega^2 r_0}{2g}, \tag{3.13}$$

and the criterion $R^* > R_c^*$ for the instability of the symmetric state can be written in the form,

$$\frac{\Delta\rho}{\rho\Omega^2} > 3\frac{D}{r_0}P^{-1}\left\{\left(\frac{Pr_0}{(1+P)gL}\right)^2 2\nu\right\}^{\frac{2}{3}}, \tag{3.14}$$

where $\Delta\rho$ represents the density difference of the fluid between the side walls. The parameter $\Delta\rho/\rho\Omega^2$ has been introduced by Hide as one of the experimentally relevant parameters. The data obtained by Fowlis & Hide (1965), and plotted in terms of this parameter, show in fact that the transition from the lower symmetric régime becomes independent of Ω for sufficiently high values of Ω. The evaluation of the right side of (3.14) yields $6{\cdot}4 \times 10^{-3}$ when the parameters of the experiment are inserted. The measured value of the transiton is $11{\cdot}8 \times 10^{-3}$. Because of the constraint of the side walls and the dissipation in the Ekman layer, the symmetric régime is more stable than suggested by the theory. For the same reason, the observed value of the wave-number is only about half the predicted value (3.10).

4. Thermal instability of a rotating heated fluid sphere

Among the problems of thermal instability the case of the self-gravitating uniformily heated, rotating fluid sphere has particular geophysical relevance. It can serve as a model of the earth core in which, hypothetically, enough radioactive material is homogeneously distributed to cause a temperature gradient which exceeds the adiabatic lapse rate. Earlier studies of the problem by Bisshopp & Niiler (1965) and by Roberts (1965), which have concentrated on the axisymmetric case, have been extended recently by Roberts (1968) to the general non-axisymmetric case. The justification for the following analysis of the problem is the relative simplicity of the perturbation approach, and the opportunity for a comparison with exact numerical results.

Although at first sight the problem seems to be quite different from the annulus problem, the mathematial discussion will in fact turn out as an application of the analysis in §3. The component of the gravity force perpendicular to the axis of rotation replaces the centrifugal force, which is negligible in the earth's core as far as the present problem is concerned. Because the distribution of the heat sources and the properties of the fluid are homogeneous, the equations allow a static solution with the temperature gradient and gravity described by $-\beta\mathbf{r}$ and $-g\mathbf{r}$ respectively. $\mathbf{r}$ is the position vector relative to the centre of the sphere. We use the radius r_0 of the sphere as length scale and $\beta r_0^3\Omega/\kappa$ as scale for the temperature. In the dimensionless description, a disturbance of the static state is governed by the equations,

$$E\nabla^2\mathbf{v} - \nabla p + \hat{R}\mathbf{r}\vartheta = 2\mathbf{k}\times\mathbf{v} + (\partial/\partial t)\mathbf{v}, \tag{4.1}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{4.2}$$

$$\nabla^2\vartheta + \mathbf{r}\cdot\mathbf{v} = \frac{\Omega r_0^2}{\kappa}\frac{\partial}{\partial t}\vartheta, \tag{4.3}$$

where the Rayleigh parameter $\hat{R}$ is defined by

$$\hat{R} = \frac{g\alpha\beta r_0^2}{\kappa\Omega}. \tag{4.4}$$

The constraining effect of the change in depth in the direction of the axis of rotation, as well as the gravity force and the basic temperature gradient, depend on the distance from the axis. This spatial dependence does not complicate the problem, however, because a mathematical analysis in terms of a 'small gap' approximation can be used. Although the small gap geometry is not enforced by side walls, as in the annulus problem, it will notwithstanding be realized, owing to the fact that the scale of the instability is small compared with 1, as in the case treated in §3(ii).

The critical Rayleigh parameter $\hat{R}_c$, which corresponds to ηR_c^*, depends, according to (3.11), on the $\frac{4}{3}$rd power of the change in depth. Since the latter increases less rapidly than the buoyancy force at small distances from the axis, and more rapidly at larger ones, we expect that the lowest value of the Rayleigh parameter will be attained at some finite distance from the axis. We assume, for the moment, that the change in depth at this distance is still small compared with 1, so that the analysis of §3 can be applied directly. We denote the distance from the axis by $\sin\theta$, and obtain in place of equation (3.7), after multiplication by η,

$$(\Delta_2 - i\hat{\omega}E^{-1}P)\left\{(E\Delta_2 - i\hat{\omega})\Delta_2 + 4\frac{\sin\theta}{2\cos^2\theta}\frac{\partial}{\partial y}\right\}p_0 = \hat{R}\sin^2\theta\frac{\partial^2}{\partial y^2}p_0. \tag{4.5}$$

We have taken into account the fact that the actual depth of the co-axial annulus in the sphere is $2\cos\theta$, instead of 1 as before, and that the surface area exceeds its projection on the x, y-plane by $\cos^{-1}\theta$. p_0 is assumed in the form

$$p_0 = f(x)\exp[i(ay + \hat{\omega}t)]. \tag{4.6}$$

Neglect of the x-dependence in (4.5) then leads to the following relations by analogy to (3.10), (3.11), and (3.8):

$$a_0 = \left(\frac{2\sin\theta P}{E(1+P)\sqrt{2}\cos^2\theta}\right)^{\frac{1}{3}}, \tag{4.7}$$

$$\hat{R}_0 = 3\left(\frac{\sqrt{2}\sin\theta P}{(1+P)\cos^2\theta}\right)^{\frac{4}{3}} E^{-\frac{1}{3}}\sin^{-2}\theta, \tag{4.8}$$

$$\hat{\omega}_0 = \left(\frac{4\sin^2\theta E\sqrt{2}}{P(1+P)^2\cos^4\theta}\right)^{\frac{1}{3}}. \tag{4.9}$$

We have used the index 0 instead of c to indicate that the critical value of the Rayleigh parameter still has to be determined by minimizing (4.8) with respect to $\sin\theta$. The maximum of $\sin\theta\cos^4\theta$ occurs at $\sin\theta_c = 1/\sqrt{5}$ corresponding to an angle

$$\theta_c \approx 26{\cdot}6^\circ. \tag{4.10}$$

Hence, the instability will first set in close to a cylindrical surface, which intersects the sphere at a latitude of about 63°. The critical values of (4.7)–(4.9) are

$$a_c = \left(\frac{P\sqrt{5/2}}{2E(1+P)}\right)^{\frac{1}{3}}, \tag{4.11}$$

$$E^{-1}\hat{R}_c = \left(\frac{2P}{E(1+P)}\right)^{\frac{4}{3}} 3(\tfrac{5}{4})^{\frac{5}{3}}, \tag{4.12}$$

$$\hat{\omega}_c = -\left(\frac{5E}{\sqrt{8}P(1+P)^2}\right)^{\frac{1}{3}}. \tag{4.13}$$

To illustrate the result a qualitative sketch of the convective motions has been drawn in figure 1.

The change in depth at the critical latitude is not really small compared with 1, and the assumption on which this analysis was based is not well satisfied. Yet, in the analogous problem of shear flow instabilities (Busse 1968), it was found in a special case that the result of the perturbation theory agrees with the exact result to within a few percent for a change in depth corresponding to (4.10). The present problem, too, offers the opportunity for a comparison with the result of an exact treatment of the equations. We shall find that (4.11)–(4.13) yield a reasonable approximation of the numerical results.

We have mentioned before that a comprehensive solution for the problem of the thermal instability in a rotating sphere was obtained by Roberts (1968); the results stated above resemble Roberts's results. The dependence on the Prandtl number is qualitatively the same and the characteristic property of (4.10) that the critical latitude is independent of the Prandtl number is closely approached by the numerical results. Serious discrepancies are found when the critical Rayleigh numbers and the critical wave-numbers are compared quantitatively. The origin for this discrepancy can be traced to the fact that it has been assumed by Roberts that the most unstable mode corresponds to a disturbance of the temperature field which vanishes in the equatorial plane. It seems unlikely on physical grounds that a mode with this property should yield the critical Rayleigh number unless the analysis is restricted to the axisymmetric case. For this reason Roberts's analysis is repeated in § 5 for modes which have the same symmetry as the perturbation solution.

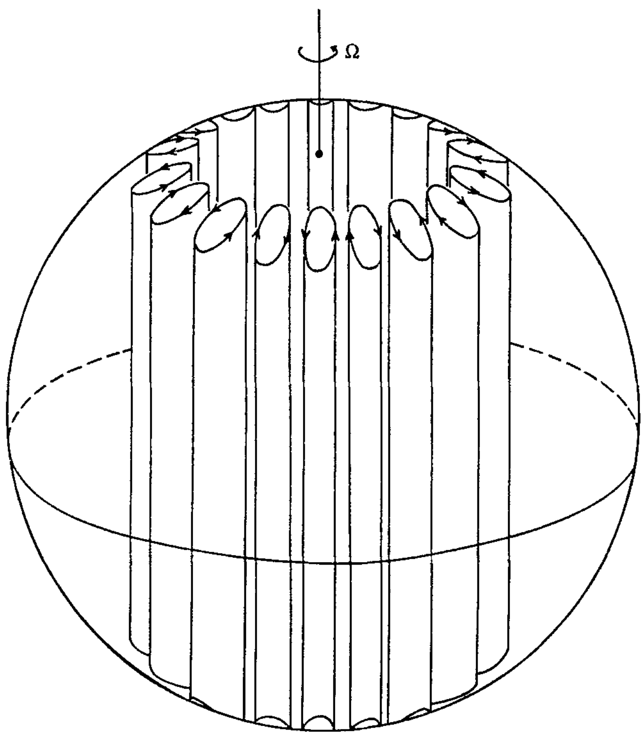

FIGURE 1. Qualitative sketch of the marginally unstable convective motions in an internally heated rotating sphere.

5. Comparison with the general asymptotic theory

In the case of a finite change in depth, the velocity component parallel to the axis of rotation is no longer negligible in comparison with the perpendicular components. In order to analyse the basic equations (4.1)–(4.3), we introduce, as a representation of the solenoidal vector field $\mathbf{v}$,

$$\mathbf{v} = \nabla \times (\nabla \times \mathbf{k}v) + \nabla \times \mathbf{k}w.$$

The elimination of the pressure from (4.1) yields the following equations for v, w and ϑ, equivalent to the system (4.1)–(4.3):

$$\left.\begin{aligned}(E\nabla^2 - i\hat{\omega})\nabla^2\Delta_2 v - 2\mathbf{k}\,.\,\nabla\Delta_2 w - \hat{R}\mathbf{k}\,.\,(\mathbf{r}\nabla^2\vartheta - \nabla(\mathbf{r}\,.\,\nabla + 1)\vartheta) &= 0,\\ (E\nabla^2 - i\hat{\omega})\Delta_2 w + 2\mathbf{k}\,.\,\nabla\Delta_2 v + \hat{R}\mathbf{k}\times\mathbf{r}\,.\,\nabla\vartheta &= 0,\\ (\nabla^2 - PE^{-1}i\hat{\omega})\vartheta - \mathbf{r}\,.\,\nabla\times(\mathbf{k}\times\nabla)v + \mathbf{k}\times\mathbf{r}\,.\,\nabla w &= 0.\end{aligned}\right\} \quad (5.1)$$

As before, the operator Δ_2 is used as abbreviation for the operator $\nabla^2 - (\mathbf{k}\nabla^2)$. In the limit of high rotation rates, it can be assumed that the characteristic scale of the solution in the ρ- and ϕ-directions of a cylindrical system of co-ordinates

(ρ, ϕ, z) is small compared with the scale in the z-direction parallel to the axis of rotation. Following Roberts (1968) analysis, we assume a solution of the form,

$$v = \exp(im\phi + i\hat{\omega}t) J_m(a\rho) F(z).$$

Accordingly, the operator Δ_2 in (5.1) can be replaced by the constant $-a^2$. The elimination of w and ϑ yields as equation for $F(z)$

$$\frac{d^2}{dz^2}F = \left\{\tfrac{1}{4}(Ea^2+i\hat{\omega})^2a^2 - \frac{\hat{R}}{4(a^2+PE^{-1}i\hat{\omega})}[(Ea^2+i\hat{\omega})(m^2+a^2z^2)-2im]\right\}F \tag{5.2}$$

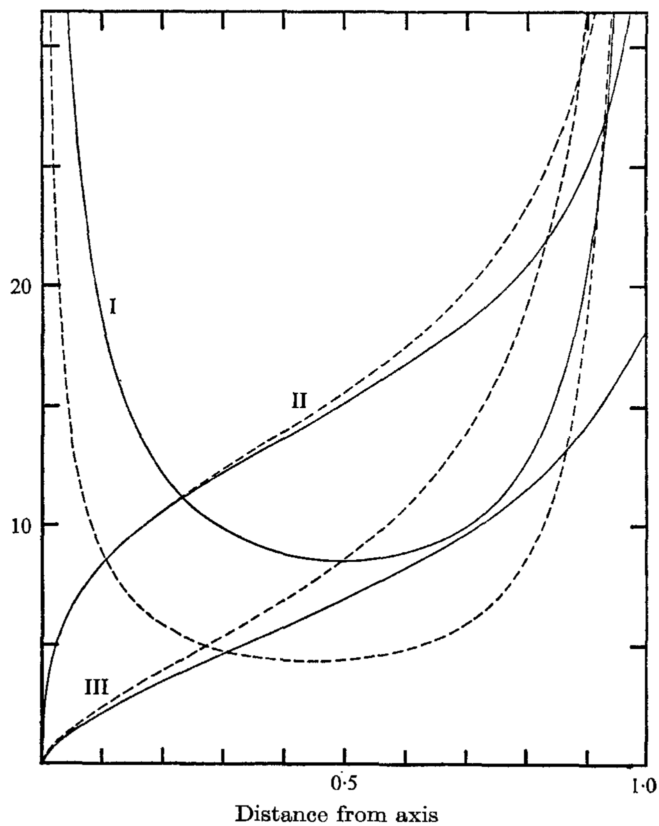

FIGURE 2. The dependence of the Rayleigh number $\hat{R}$, the wave-number a, and the frequency $\hat{\omega}$ on the distance from the axis of rotation in the case $P = 1$. Curves I, II, III describe $\hat{R}E^{\frac{1}{3}}$, $2aE^{\frac{1}{3}}$, and $-\hat{\omega}E^{-\frac{1}{3}}$ respectively. The dashed lines correspond to (4.8), (4.7), (4.9).

The boundary condition $\mathbf{v}.\mathbf{r} = 0$ on the surface of the sphere can be expressed by

$$E(a^2+i\omega)a^2zF + im2F' = 0 \quad \text{for} \quad z = (1-\rho^2)^{\frac{1}{2}}. \tag{5.3}$$

Here, the fact has been used that for large m, $J_m(a\rho)/J_m(m)$ differs from zero essentially only in the neighbourhood of $\rho = m/a$. The eigenvalue problem given by (5.2), (5.3) is identical with the problem described by (7.10)–(7.12) and (7.23) in Roberts (1968). The problem has solutions $F(z)$, which are either symmetric or antisymmetric with respect to the z-dependence. Roberts assumed that the lowest

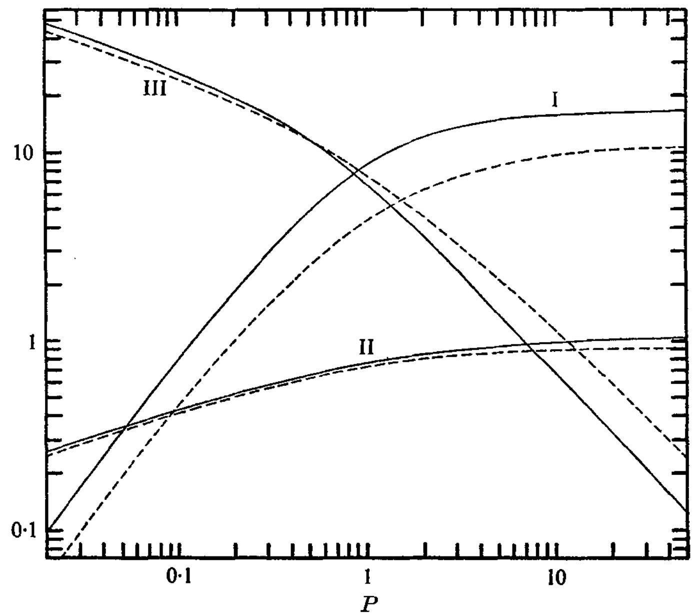

FIGURE 3. Dependence of the critical Rayleigh number, wave-number, and frequency upon the Prandtl number. Curves I, II, III describe $\hat{R}_c E^{\frac{1}{3}}$, $a_c E^{\frac{1}{3}}$, and $-\hat{\omega}_c E^{-\frac{1}{3}}$ respectively. The dashed curves correspond to (4.11), (4.12), (4.13).

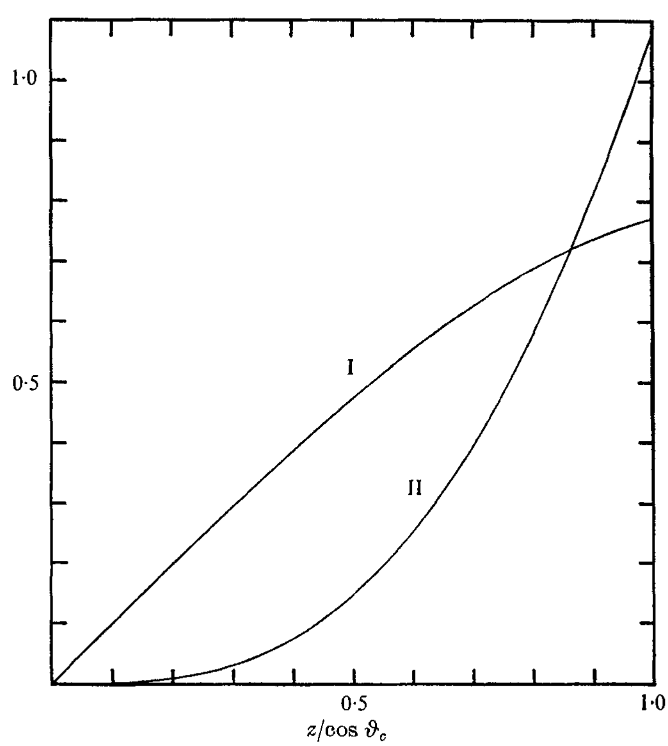

FIGURE 4. The real part (I) and the imaginary part (multiplied by 10) of $F(z/\cos\theta_c)$ for $\hat{R} = \hat{R}_c$ are plotted in the case $P = 1$.

value of the Rayleigh parameter corresponds to the symmetric case. The following results, however, obtained for an antisymmetric function $F(z)$, yield a lower eigenvalue $\hat{R}$. In figure 2 the minimum of $\hat{R}$ with respect to the dependence on a has been plotted as a function of $\rho = \sin\theta$ in the case $P = 1$. A comparison with (4.8) shows that the numerical result exceeds the result of the perturbation theory by a factor of about two. This discrepancy is caused by the fact that the conditions, under which the perturbation theory has been derived, are not satisfied, even for small values of ρ, when the change in depth in the z-direction becomes small. The component of the buoyancy force perpendicular to the axis of rotation vanishes in this case like ρ^2, while the parallel component remains of order 1. This fact violates the condition under which the perturbation theory has been derived, since the ratio between the perpendicular and the parallel component should be large compared to the perturbation parameter ρ.

P	$a_c(E/2)^{\frac{1}{3}}$	$\hat{R}_c(E/16)^{\frac{1}{3}}$	$\sin\theta_c$	$\hat{\omega}_c(4E)^{-\frac{1}{3}}$
0·025	0·223	0·0525	0·519	−2·84
0·1	0·343	0·3151	0·513	−1·65
0·3	0·465	0·149	0·508	−1·008
0·6	0·544	2·283	0·5005	−0·645
1	0·6003	3·382	0·5004	−0·4362
1·5	0·643	4·253	0·504	−0·303
3	0·709	5·396	0·521	−0·1513
10	0·785	6·265	0·550	−0·0423
40	0·817	6·520	0·565	−0·0100

TABLE 1. Dependence of the critical parameters on the Prandtl number

The rather close agreement, however, between the approximate expressions (4.7) and (4.9) for the wave-number and a frequency, and the numerical results, suggests that the perturbation theory yields the correct dynamic description of the convection mode. This aspect is emphasized in figure 3, in which the critical values $\hat{\omega}_c$, a_c, and $\hat{R}_c$, corresponding to the absolute minimum of $\hat{R}$, have been plotted as functions of the Prandtl number. Numerical values are given in table 1 in a form in which they can be directly compared with the results for the even mode given by Roberts. The comparison shows that the critical Rayleigh numbers for the first even mode exceed the values of the first odd mode by a factor of about 4. It is worth mentioning that the even mode propagates eastward, as well as the odd mode, according to the negative sign of $\hat{\omega}$, which was not indicated by Roberts. The critical latitude, at which the Rayleigh number reaches its critical value, is nearly independent of P according to table 1, in close agreement with the value (4.10).

The perturbation results can be derived directly from (5.1), if w is assumed z-independent, and if v is assumed proportional to z. The coefficient in the latter case has to be determined by the boundary condition. A nearly linear dependence of $F(z)$ is exhibited by the numerical computations as shown in figure 4 for the critical mode in the case $P = 1$.

Although the perturbation results underestimate the Rayleigh number, the agreement with the numerical results is remarkable, considering the simple derivation of (4.7)–(4.13). This suggests that the perturbation theory has a wider range of application than can be expected from its derivation. Physically, the process described by the perturbation theory can be interpreted as the stretching and compressing of vortex lines parallel to the axis of rotation, as in the theory of Rossby waves. Filaments changing their distance from the axis acquire vorticity relative to the rotating system. The interaction of the vortices then causes the eastward motion of the convection pattern. The qualitative agreement with the exact treatment indicates that this picture gives the correct description for the dynamical constraint governing convection in a rotating fluid sphere.

6. Concluding remarks

A common feature of the problems discussed here is the existence of a temperature gradient and a component of the buoyancy force in the direction perpendicular to the axis of rotation. From this point of view, the well-known Bénard problem of a horizontal fluid layer heated from below, and rotating about a vertical axis, appears as a somewhat singular case. The perturbation method employed here fails in this particular case, because the buoyancy force coincides with the axis of rotation. Viscous dissipation becomes necessary for the onset of stationary convection and enters the basic balance in order to overcome the constraint of the Taylor–Proudman theorem. A detailed analysis of the problem is given in Chandrasekhar's (1961) treatise. In the problem of the heated fluid sphere considered in §§4 and 5 both kinds of instability mechanism are possible. Roberts (1968) has shown that the critical Rayleigh number for the 'Bénard' mechanism exceeds the value (4.12). He has also shown that the same fact holds for instabilities in the form of inertial oscillations which have lower critical Rayleigh numbers than the stationary axisymmetric instabilities at sufficient low Prandtl numbers.

A particular property of the instabilities in the form of Rossby waves in contrast to the stationary modes considered in §2 is the fact that their non-linear interaction can generate a mean flow in the form of a differential rotation. This finite amplitude property is of importance for astrophysical and geophysical applications and will need special consideration. The relation problem of differential rotation induced by convection in a spherical shell has been discussed in another paper (Busse 1970).

The effect of rotation on buoyancy driven instabilities is always stabilizing in contrast to shear flow instabilities, for instance. The analysis of §§2–5 has shown the inhibiting effect of the viscous dissipation in the Ekman layers and of the change in depth of the system in the direction of the axis of rotation. The viscous dissipation in the interior plays a destabilizing role, since it permits the convection to balance the constraint of the changing depth. A detailed experimental investigation of the problem is still lacking, although the use of the centrifugal force offers a convenient possibility for a buoyancy force in a rotating system. Experiments with an rotating annulus would, in addition, have the advantage of providing a model for convection in the earth's core, owing to the close analogy between the two problems suggested by the theory.

The author gratefully acknowledges the assistance of Mrs Trostel in the numerical computations.

Appendix. Disturbances in the form of inertial oscillations

In constrast to the previous analysis, we consider in this appendix disturbances governed by

$$\left.\begin{aligned} 2\mathbf{k}\times\mathbf{v}_0+\nabla p_0+i\lambda_0\mathbf{v}_0 &= 0, \\ \nabla.\mathbf{v}_0 &= 0, \end{aligned}\right\} \tag{A 1}$$

as basic balance in place of (2.15). λ_0 is a real number representing the frequency of oscillation. The boundary condition,

$$\mathbf{n}.\mathbf{v}_0 = 0 \tag{A 2}$$

on the surfaces $z = \pm\frac{1}{2}$ and $x = \pm\frac{1}{2}D/L$, imposes a strong constraint, with the consequence that the manifold of solutions of (A 1) is discrete with respect to the x-dependence as well as to the z-dependence. It is shown in Greenspan (1968) that the velocity vector $\mathbf{v}_0$ can be expressed by the pressure,

$$(1-\tfrac{1}{4}\lambda_0^2)\mathbf{v}_0 = \tfrac{1}{2}\mathbf{k}\times\nabla p_0 - i\frac{\lambda_0}{4}\nabla p_0+\frac{i}{\lambda_0}\mathbf{k}(\mathbf{k}.\nabla p_0). \tag{A 3}$$

Equations (A 1), together with the boundary condition (A 2), yield the Poincaré eigenvalue problem in terms of the pressure,

$$\Delta p_0 - \frac{4}{\lambda_0^2}(\mathbf{k}.\nabla)^2 p_0 = 0, \tag{A 4}$$

with

$$\frac{\partial}{\partial z}p_0 = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2} \tag{A 5}$$

and

$$-\lambda_0\frac{\partial}{\partial x}p_0 + 2i\frac{\partial}{\partial y}p_0 = 0 \quad \text{at} \quad x = \pm\frac{1}{2}\frac{D}{L} \tag{A 6}$$

as boundary conditions.Equation (A 4) admits solutions of the form,

$$p_0 = \exp[i(ay+\lambda_0 t)]\cos n\pi(z+\tfrac{1}{2})\begin{Bmatrix}\cos\mu_m x\\ \sin\mu_m x\end{Bmatrix}, \tag{A 7}$$

where the upper function corresponds to an odd integer m and the lower function to an even integer m.

The condition (A 5) is satisfied by this class of solutions, and (A 6) yields the following equations for the determination of μ_m:

$$\lambda_0\mu_m\sin\mu_m\frac{D}{2L} - 2a\cos\mu_m\frac{D}{2L} = 0 \quad \text{for} \quad \text{odd } m, \tag{A 8a}$$

$$-\lambda_0\mu_m\cos\mu_m\frac{D}{2L} - 2a\sin\mu_m\frac{D}{2L} = 0 \quad \text{for} \quad \text{even } m. \tag{A 8b}$$

The frequency λ_0 is a function of the integers m and n, and of the wave-number a, and varies between -2 and $+2$:

$$\lambda_0 = \pm 2n\pi/[(n\pi)^2+a^2+\mu_m^2]^{\frac{1}{2}}. \tag{A 9}$$

The equation of order $E^{\frac{1}{2}}$ for $\mathbf{v}_1$ has the following form:

$$2\mathbf{k}\times\mathbf{v}_1+\nabla p_1+i\lambda_0\mathbf{v}_1 = -R\left(\mathbf{i}-\mathbf{k}\frac{g}{\Omega^2 r_0}\right)\vartheta_0 - E^{-\frac{1}{2}}(\mathbf{U}.\nabla\mathbf{v}_0+\mathbf{v}_0.\nabla\mathbf{U})-i\lambda_1\mathbf{v}_0. \tag{A 10}$$

For the frequency λ an expansion analogous to (2.11) has been assumed. The critical value of the Rayleigh parameter for the onset of oscillations is determined as the lowest value of R for which solutions of (A 10) are possible with a real value of λ_1. By multiplying (A 10) with v_0^+, which is the complex conjugate of v_0, and subsequent integration, the following solvability condition for (A 10) is obtained:

$$\oint p_0^+\mathbf{n}.\mathbf{v}_1 d\Sigma = \int v_0^+\left\{\left(\mathbf{i}-\frac{g}{\Omega^2 r_0}\mathbf{k}\right)R\vartheta_0 - E^{-\frac{1}{2}}\left(\mathbf{v}_0.\mathbf{k}\frac{d}{dz}\mathbf{U}+\mathbf{U}.\nabla\mathbf{v}_0\right)-i\lambda_1\mathbf{v}_0\right\}dV. \tag{A 11}$$

We have used the fact that $\mathbf{v}_0^+$ satisfies (A 1) with $-\lambda_0$ in place of λ_0. The left-hand side of (A 11) is determined by the inhomogeneous boundary condition for $\mathbf{n}.\mathbf{v}_1$ which, according to Greenspan (1968, formula (2.9.14)), leads to a contribution only at the vertical side walls in the problem considered here:

$$\oint p_0^+\mathbf{n}.\mathbf{v}_1 d\Sigma = 2^{\frac{1}{2}}\iint |\mathbf{k}.\mathbf{v}_0|^2|\lambda_0|^{\frac{1}{2}}(1+i\lambda_0/|\lambda_0|)\,dy\,dz|_{x=D/2L}. \tag{A 12}$$

On the right-hand side of (A 11) a number of terms vanish. The z-integration over such terms as $z\mathbf{v}_0^+.\mathbf{v}_0$ and $\mathbf{v}_0.\mathbf{k}\mathbf{k}\times\mathbf{v}_0^+$ gives zero, because the vertical and the horizontal component of the velocity field have opposite symmetry with the respect to the z-dependence, according to (A 3) and (A 7). The heat equation,

$$\Delta\vartheta_0 = \mathbf{i}.\mathbf{v}_0+i\lambda_0\frac{\Omega L^2}{\kappa}\vartheta_0, \tag{A 13}$$

shows that ϑ_0 has the same symmetry as the x-component of the velocity. Hence, the contribution of the gravity term, $\mathbf{k}.\mathbf{v}_0^+\vartheta_0$, vanishes as well. The solution of (A 13) satisfying the boundary conditions (2.6) is given, according to (A 7), by

$$\vartheta_0 = -\mathbf{i}.\mathbf{v}_0\left((\pi n)^2+a^2+\mu_m^2+i\lambda_0\frac{\Omega L^2}{\kappa}\right)^{-1}. \tag{A 14}$$

The evaluation of the imaginary parts of (A 11) and (A 12) lead to a relation for λ_1 which is not of interest at the moment. The evaluation of the real parts yields the following expression for R:

$$R = \frac{(n\pi)^2(1-\frac{1}{4}\lambda_0^2)^2\left[((n\pi)^2+a^2+\mu_m^2)^2+\left(\lambda_0\frac{\Omega L^2}{\kappa}\right)^2\right]\left(1\pm\cos\frac{D}{L}\mu_m\right)\sqrt{2}}{|\lambda_0|^{\frac{3}{2}}\frac{D}{4L}a^2((n\pi)^2+a^2+\mu_m^2)\left(1\pm\frac{L}{\mu_m D}\sin\frac{D}{L}\mu_m+\frac{\lambda_0^2\mu_m^2}{4a^2}\left(1\mp\frac{L}{\mu_m D}\sin\frac{D}{L}\mu_m\right)\right)}, \tag{A 15}$$

where the upper sign corresponds to an odd, the lower sign to an even, integer m. Using (A 9) we can rewrite relation (A 15) as

$$R = \frac{[(\pi n)^2 + a^2 + \mu_m^2]^2 + (\lambda_0 E^{-1}P)^2}{a^2} \frac{n\pi \dfrac{2L}{D}\left(\dfrac{2}{|\lambda_0|}\right)^{\frac{1}{2}} (1-\lambda_0^2/4)^2 \left(1 \pm \cos \dfrac{D}{L}\mu_m\right)}{1 \pm \dfrac{L}{\mu_m D}\sin\dfrac{D}{L}\mu_m + \dfrac{\lambda_0^2 \mu_m^2}{4a^2}\left(1 \mp \dfrac{L}{\mu_m D}\sin\dfrac{D}{L}\mu_m\right)}. \tag{A 16}$$

The first term on the right-hand side resembles (2.26) in the case when the Prandtl number is set to zero. In most physical situations, $E^{-1}P$ will be large compared with one, and the expression (A 16) will exceed the critical value (2.27) for any choice of the parameter n, m, and a. Only if L/D and the Prandtl number are so small that $E^{-2}P^2L/D$ becomes of the order unity may (A 16) yield lower values than (2.27).

In this case, however, the stabilizing influence of the interior dissipation will not be negligible. Hence, we conclude that for all practical purposes the disturbances in the form of inertial oscillations can be disregarded as possible modes of instability.

REFERENCES

Bisshopp, F. E. & Niiler, P. P. 1965 *J. Fluid Mech.* **23**, 459.
Busse, F. H. 1968 *J. Fluid Mech.* **33**, 577.
Busse, F. H. 1970 *Astrophys. J.* **159**, 629.
Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford: Clarendon.
Fowlis, W. W. & Hide, R. 1965 *J. Atmos. Sci.* **22**, 541.
Fultz, O, 1961 *Adv. Geophysics*, **7**, 1.
Greenspan, H. P. 1968 *The Theory of Rotating Fluids*. Cambridge University Press.
Hide, R. 1958 *Phil. Trans.* A **250**, 441.
Hunter, C. 1967 *J. Fluid Mech.* **27**, 753.
Roberts, P. H. 1965 *Astrophys. J.* **141**, 240.
Roberts, P. H. 1968 *Phil. Trans.* A **263**, 93.
Robinson, A. R. 1959 *J. Fluid Mech.* **6**, 599.